JC13 Rec'd PCT/PTO 15 MAR 2001

FORM PTO-1390 U S DEPARTMENT OF COMMERCE PATENT AND TRADEMARK OFFICE		ATTORNEY'S DOCKET NUMBER
TRANSMITTAL LETTER TO THE UNITED STATES DESIGNATED/ELECTED OFFICE (DO/EO/US) CONCERNING A FILING UNDER 35 U.S.C. 371		52254-016
		U.S. APPLIC. NO. (if known, see 37 CFR 1.5) 09/787290
INTERNATIONAL APPLICATION NO. PCT/US99/21955	INTERNATIONAL FILING DATE September 22, 1999	PRIORITY DATE CLAIMED September 22, 1998
TITLE OF INVENTION DEVICES AND TECHNIQUES FOR LOGICAL PROCESSING		
APPLICANT(S) FOR DO/EO/US Jonathan WESTPHAL		

OIPE JC147 MAR 15 2001 PATENT & TRADEMARK OFFICE

Applicant herewith submits to the United States Designated/Elected Office (DO/EO/US) the following items and other information:

1. ☒ This is a **FIRST** submission of items concerning a filing under 35 U.S.C. 371.
2. ☐ This is a **SECOND** or **SUBSEQUENT** submission of items concerning a filing under 35 U.S.C. 371.
3. ☒ This express request to begin national examination procedures (35 U.S.C. 371(f)) at any time rather than delay examination until the expiration of the applicable time limit set in 35 U.S.C. 371(b) and PCT Articles 22 and 39(1).
4. ☒ A proper Demand for International Preliminary Examination was made by the 19th month from the earliest claimed priority date.
5. ☒ A copy of the International Application as filed (35 U.S.C. 371(c)(2))
   a. ☐ is transmitted herewith (required only if not transmitted by the International Bureau).
   b. ☒ has been transmitted by the International Bureau (A copy of the published application is transmitted herewith).
   c. ☐ is not required, as the application was filed in the United States Receiving Office (RO/US)
6. ☐ A translation of the International Application into English (35 U.S.C. 371(c)(2)).
7. ☐ Amendments to the claims of the International Application under PCT Article 19 (35 U.S.C. 371(c)(3))
   a. ☐ are transmitted herewith (required only if not transmitted by the International Bureau).
   b. ☐ have been transmitted by the International Bureau.
   c. ☐ have not been made; however, the time limit for making such amendment has NOT expired.
   d. ☐ have not been made and will not be made.
8. ☐ A translation of the amendments to the claims under PCT Article 19 (35 U.S.C. 371(c)(3))
9. ☐ An oath or declaration of the inventor(s) (35 U.S.C. 371(c)(4)).
10. ☐ A translation of the annexes to the International Preliminary Examination Report under PCT Article 36 (35 U.S.C. 371(c)(5)).

Items 11. to 16. below concern other document(s) or information included:

11. ☒ An Information Disclosure Statement under 37 CFR 1.97 and 1.98.
12. ☐ An assignment document for recording. A separate cover sheet in compliance with 37 CFR 3.28 and 3.31 is included.
13. ☒ A FIRST preliminary amendment.
    ☐ A SECOND or SUBSEQUENT preliminary amendment.
14. ☐ A substitute specification.
15. ☐ A change of power of attorney and/or address letter.
16. ☒ Other items or information.

International Search Report
International Preliminary Examination Report
Forms PCT/IB/304, PCT/IB/306, and PCT/IB/308

20277
PATENT TRADEMARK OFFICE

JC03 Rec'd PCT/PTO 15 MAR 2001

U.S. APPLIC. NO. (if known, see 37 CFR 1.50)	INTERNATIONAL APPLICATION NO.	ATTORNEY'S DOCKET NUMBER
09/787290	PCT/US99/21955	52254-016

				CALCULATIONS	PTO USE ONLY
17. ☒ The following fees are submitted: **Basic National Fee (37 CFR 1.492(a)(1)-(5)):** Search Report has been prepared by the EPO or JPO $860.00 International preliminary examination fee paid to USPTO (37 CFR 1.482) $690.00 No international preliminary examination fee paid to USPTO (37 CFR 1.482) but international search fee paid to USPTO (37 CFR 1.445(a)(2)) $710.00 Neither international preliminary examination fee (37 CFR 1.482) nor international search fee (37 CFR 1.445(a)(2)) paid to USPTO $1,000.00 International preliminary examination fee paid to USPTO (37 CFR 1.482) and all claims satisfied provisions of PCT Article 33(2)-(4) $100.00					
ENTER APPROPRIATE BASIC FEE AMOUNT =				$ 690.00	
Surcharge of $130.00 for furnishing the oath or declaration later than ☐ 20 ☒ 30 months from the earliest claimed priority date (37 CFR 1.492(e)).				$ 130.00	
Claims	Number Filed	Number Extra	Rate		
Total Claims	12 -20 =	0	x $18.00	$	
Independent Claims	7 -3 =	4	x $80.00	$ 320.00	
Multiple dependent claim(s) (if applicable)			+ $270.00	$	
TOTAL OF ABOVE CALCULATIONS =				$ 1,140.00	
Reduction by 1/2 for filing by small entity, if applicable. Verified Small Entity Statement must also be filed. (Note 37 CFR 1.9, 1.27, 1.28).				$	
SUBTOTAL =				$ 1,140.00	
Processing fee of $130.00 for furnishing the English translation later than the ☐ 20 ☐ 30 months from the earliest claimed priority date (37 CFR 1.492(f)). +				$	
TOTAL NATIONAL FEE =				$ 1,140.00	
Fee for recording the enclosed assignment (37 CFR 1.21(h)). The assignment must be accompanied by an appropriate cover sheet (37 CFR 3.28, 3.31). $40.00 per property +				$	
TOTAL FEES ENCLOSED =				$ 1,140.00	
				Amount to be: refunded	$
				charged	$

a. ☐ A check in the amount of $ _____ to cover the above fees is enclosed

b. ☒ Please charge my Deposit Account No. 500417 in the amount of $ 1,140.00 to cover the above fees. A duplicate copy of this sheet is enclosed.

c. ☒ The Commissioner is hereby authorized to charge any additional fees which may be required, or credit any overpayment to Deposit Account No. 500417. A duplicate copy of this sheet is enclosed.

NOTE: Where an appropriate time limit under 37 CFR 1.494 or 1.495 has not been met, a petition to revive (37 CFR 1.137(a) or (b)) must be filed and granted to restore the application to pending status.

SEND ALL CORRESPONDENCE TO:	
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BASED ON FORM PTO-1390 (Rev. 5-93)

Docket No.: 52254-016 PATENT

IN THE UNITED STATES PATENT AND TRADEMARK OFFICE

In re Application of	:	
Jonathan WESTPHAL	:	
Serial No.:	:	Group Art Unit:
Filed: March 15, 2001	:	Examiner:

For: DEVICES AND TECHNIQUES FOR LOGICAL PROCESSING

PRELIMINARY AMENDMENT

Commissioner for Patents
Washington, DC 20231

Sir:

Prior to examination of the above-referenced application, please amend the application as follows:

IN THE SPECIFICATION:

Page 3, line 6, please change "Figure 18B is an illustration of an example of the Fix Rule." to --Figures 18B and 18C are illustrations of an example of the Fix Rule.--

Page 3, line 24, please change "Figure 31 is an illustration of interferometric processing for modus ponens to the", to "Figure 31 is an illustration of interferometric processing for modus ponens.--

REMARKS

Entry of this preliminary amendment is respectfully requested.

Respectfully submitted,

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09/787290
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DEVICES AND TECHNIQUES FOR LOGICAL PROCESSING

BACKGROUND OF THE INVENTION

Field of the invention

This invention relates to the field of logical processing and, more particularly to devices and techniques for simplifying digital logic.

Description of related art

Logic can be described as techniques and operations by which one moves from what one knows to be true to new truths. The principles of logic have been applied in the design and operation of digital logic circuits. Modern-day computers and other processing devices have utilized digital logic extensively. Many of the problems to which digital logic can be applied are complex, involving many independent variables. This results in extremely complex logical circuits in which large numbers of operations are performed. The cost associated with manufacturing and fabrication of such complex digital circuits is great. It would be highly desirable to reduce the number of complements required for performing a particular logical function or set of functions while at the same time increasing the speed with which those functions can be performed.

Digital computers are, of course, well-known. More recently, optical computers have been developed which can perform logical functions using optical elements. These optical computers can perform the same functions performed by digital computers but in principle much faster.

SUMMARY OF THE INVENTION

The invention is directed to apparatus, methods, systems and computer program products which permit a simplification of the logic required for performing a certain function to a minimum set of logical elements of operations. The this permits the complexity of digital circuitry to be simplified the processing speed with which complex digital operations can be performed, reduced.

This is accomplished using a system of propositional logic in which propositions are represented as vectors or displacement in a space. This is applied to the simplification problem, the problem of finding a method for reducing logical schema to a shortest

equivalent. Applications of these techniques to the problems of electrical circuit minimization, to free-space optical processing, to flat optical processing, and to logical processing using color imagery are described.

BRIEF DESCRIPTION OF DRAWINGS

The objects, features and advantages of the system of the present invention will be apparent from the following description in which:

Figure 1 represents two-dimensional space for propositions.

Figure 2 illustrates the propositions p and $p \vee q$ in the space of figure 1.

Figure 3 is a diagram of the vector two-dimensional space showing in the conditional normal schemata or CNS-plane.

Figure 4 is a diagram of the vector two-dimensional space showing the alternational normal schemata or the ANS- plane.

Figure 5 is a diagram showing modus ponens in the CNS-plane.

Figure 6 is a diagram showing modus tollens in the CNS- plane

Figure 7 use a diagram showing the disjunctive syllogism in the CNS- plane.

Figure 8 is the diagram showing how the ANS- and CNS- planes relate.

Figure 9 is a diagram illustrating operations within the ANS- space.

Figure 10 shows an extension of the ANS- plane of Figure 4 to a three-dimensional ANS-space.

Figure 11 shows an extension of the CNS- space to three dimensions together with a hypothetical syllogism.

Figure 12A illustrates a hypothetical syllogism with three variables in the CNS- space.

Figure 12B shows a view of the hypothetical syllogism in the three-dimensional CNS-space.

Figure 13A illustrates a cancellation technique used in simplifying logical representations and in accordance with the invention.

Figure 13B shows the representations of Figure 13A in graphical form.

Figure 13C illustrates implication and equivalence.

Figures 14 A, 14B and 14C illustrate a solution to the simplification problem using the techniques of the invention.

Figure 15 shows a 4-clause schema simplified.

Figure 16A shows a 3-clause schema simplified.

Figure 16B shows the truth-table for the representation of Figure 16A.

Figure 17 shows a 4-variable vector diagram simplification.

Figure 18A is a diagram illustrating the Fix Rule for d = 2.

Figure 18B is an illustration of an example of the Fix Rule.

Figure 19 is an illustration of the Fix Rule for d = 3.

Figure 20 illustrates application of the invention to situations in which developed normal formulas are not the point of departure.

Figure 21 illustrates the equivalence of a developed alternational formal and its undeveloped counterpart.

Figure 22 illustrates the simplification of an undeveloped set of statements.

Figure 23 illustrates another simplification of an undeveloped set of statement taken from Quine.

Figure 24 illustrates an equivalence within the set of statements shown in Figure 23.

Figure 24A illustrates superfluity in the Consensus Theorem and its dual in the CNS-form in a truth-table.

Figure 25 illustrates the Consensus Theorem.

Figure 26 illustrates the dual of the Consensus Theorem.

Figure 27 illustrates a superfluity shown in Figure 23.

Figure 28 illustrates a target circuit to be simplified in accordance with the invention.

Figure 29 shows a simplest circuit equivalent to the target circuit.

Figure 30 is an illustration of optical computation of modus ponens.

Figure 31 is an illustration of interferometric processing for modus ponens to the

Figure 32 illustrates an optical element used for disconjunction and conjunction in a free-space optical processing.

Figure 33 is an illustration of flat optical processing.

Figure 34 or is an illustration of vector addition utilizing sequences of spatial light modifiers.

Figure 35 is an illustration of colorimetric computation of modus ponens.

Figure 36 is a colorimetric simplification of $\mathbf{pq} \vee \mathbf{p}\overline{q}$.

DESCRIPTION OF THE PREFERRED EMBODIMENT

Part I of this paper describes a system of propositional logic in which propositions are represented as vectors or displacements in a space. Part II gives the application of the system to the simplification problem, the problem of finding a method for reducing a truth-functional schemata in alternational normal form to a shortest equivalent. Part III is about applications: (i) to problems of electrical circuit minimization; (ii) to free-space optical processing; (iii) to "flat" optical processing; and (iv) to logical processing using colorimetry.

Part I

Imagine a space in which the co-ordinates from the origin **0** are propositional addresses or *possibilities*. Let (1, 0) be the propositional address **p**, and (0,1) the propositional address **q**. Then (1, 1) is the point **p, q**, and we can let the sign "+" be an operation on **p** and **q** which is defined by distance and direction from the origin, by which '**p+q + p+q**' is an instruction to go two units in a **p**-ward direction, and two units in a **q**-ward direction. The operation performed by someone obeying this instruction is commutative and associative.

We can now represent the *proposition* that **p** as a directed line-segment or vector along the **p**- or **x**-axis from the origin **O** to the point or propositional address p in the space, representing the vector **p** in boldface as is standardly done to distinguish it from the possibility p which is represented as the point at the arrowhead of **p**. We can also let the vector **q** be the proposition **q**, represented in the space as the vector pointing straight up the **q**- or **y**-axis to the point q.

Now if we build up the space interpreting the operation "+" as "**v**", the **x**- and **y**-axes will obviously represent lines of logical equivalence. At (2, 0), or **p, p**, for example, we will find the arrowhead of **p v p,** and at (3, 0) the arrowhead of **p v p v p.** These and the rest of the proposition vectors along the **p**-axis are logically equivalent to the base

vector **p**. At (0, 2), or **q, q**, we will find the arrowhead of **q v q**. Thus **p** and **q** will stand in for the usual unit vectors **i** and **j**. (The vector space of propositions can however have infinitely many directions such as **s, t, u, v...**, which will become important later on when a technique is given to simplify propositions with large numbers of literals.)

We are also now in a position to represent the proposition **p v q** in the space as **p + q**, the vector resultant of the vectors **p** and **q**, which travels from the origin to p, **q**. Then **p v q** is itself a vector.

I will call a vector diagram for the propositional calculus such as Figure 2 a **V-Diagram** for the proposition or schema. ("V" is for vector, which is from the Latin word meaning "carrier", "traveller" or " rider".) The **V**-diagram can be further built up by adding the negation symbols for the negative vectors $\bar{p}$ and $\bar{q}$ in the negative or reverse directions along their respective axes. So we arrive at all of the literals, which are single letters and negations of single letters, and we can also find pairs of single negated or unnegated letters, the propositions **p v** $\bar{q}$, and $\bar{p}$ **v** $\bar{q}$.

Let us call the vector two-space in Figure 3 the CNS-plane for the plane of the "conjunctional normal schemata".

We are now free to explore another plane, the plane of the alternational normal schemata, or the ANS-plane as I shall call it, in which the points are not alternations but conjunctions, and the vector operation "+" within the space is interpreted as alternation. The ANS-plane and the CNS-plane are duals, so that each point in each plane correspond to its dual in the other plane. This also means that the uniting operation in the CNS-plane is related to the dual of the operation in the ANS-plane, and *vice versa*. In the CNS-plane "+" is alternation, and so in the ANS-plane it is conjunction. The operation "→" in "α→β" in the CNS-plane is to be read as implication or the assertion of the conditional. In the ANS-plane "→" is to be read as the denial of the *negation* of implication, which is the denial of the conjunction of the antecedent with the negation of the consequent. The whole ANS-plane is to be read as a systematic set of denials, the denials that the propositions given at the base of the vector arrowheads imply a contradiction. This will

be obvious if we remember that arrows *ending* at the origin rather than those issuing from it, as in the CNS-plane, are assertions in the ANS-plane.

In both of the planes certain familiar truths appear as expressions of the main principle which governs "+" or vector addition, the so-called parallelogram law of Galileo. In the CNS-plane we can think of the premises of an argument as component vectors, and the resultant as the conclusion. Then an elementary valid argument–formrin the CNS-plane is a parallelogram starting at the origin **O** in that plane. The conjunction of the alternations yields the conclusion, and we get modus ponens appearing as in Diagram 5. If the vectors are represented as displacements around **O** in the **V**-diagram, the modus ponens in the CNS-plane is the set of displacements

	p	q		
Premise 1	-1	1	+	$\bar{p} \vee q$
Premise 2	1	0		p
	———			
Conclusion=	0	1		q

Modus tollens appears as:

	p	q		
Premise 1	-1	1	+	$\bar{p} \vee q$
Premise 2	0	-1		$\bar{q}$
	———			
Conclusion=	-1	0		$\bar{p}$

The disjunctive syllogism appears, with its displacement matrix, as:

	p	**q**		
Premise 1	1	1	+	**p v q**
Premise 2	0	-1		$\overline{p}$
Conclusion=	0	1		**q**

Consider now the relation between the CNS-plane and the ANS-plane. There clearly is one, as they share the literals and the all-important origin **0**. The two planes can be brought into harmony if we represent them, arbitrarily, as lying above and below the origin in a space whose third dimension runs along the conjunction-alternation axis, putting alternation at the top and conjunction at the bottom.

The result is a space, or the part of it near **0**, with two planes above and below the origin. The origin **0** appears in the vertical axis between the two planes. The whole space of Figure 8 generates further principles of the propositional calculus. Take **p v q** in the top right hand corner. Negating it comprehensively, in all three dimensions, or developing it through the origin, gives the point $\overline{p}\ \overline{q}$ in the ANS-plane. This is one of the two forms of DeMorgan's theorem. Its other form can be found by comprehensively negating **pq** in the ANS-plane, and travelling through **0** to $\overline{p}$ **v** $\overline{q}$ in the CNS-plane. The CNS-/ANS-space as a whole has an intriguing and beautiful structure, as it combines the dimensions of alternation and conjunction, the various propositions formed from atomic **p** and **q**, and the dimension of negation.

Operations within the ANS-space have "+" representing conjunction. When all the non-equivalent conjunction points are established in the space, pairs and other combinations of the given points or conjunctions are given as alternations or vectors. So

we get a resultant of **pp** from **pq v p**$\bar{q}$ by relating the two vectors to the origin **0** in a parallelogram (Figure 9).

In the CNS-space, on the other hand, the corresponding operation produces alternations. and the operation within the space which combines them is conjunction. So we get sets of conjunctions, e.g. those important ones involving $\bar{p}$ **v q**, which are among the more important *arguments* of natural deduction.

Assume now in the ANS-space a third proposition **r**, and a third dimension **z** in which the unit vector **r** is to be found. So we get the **r**-plane, the one swept out by the vector **r**. This space can also be represented in two dimensions on the page. In Figure 10 the negation-affirmation axis, which follows the **z**-axis in the order of rotation of the variables about **0**, is inserted to prevent the occlusion of lines and points.

To check the validity of the hypothetical syllogism (-1, 1, 0) (0, -1, 1), (-1, 0, 1), with three variables, in the CNS-space, we can represent it as in Figure 11.

The validity of the argument appears, using three sets of coordinates, as

	p	**q**	**r**		
Premise 1	-1	1	0	+	$\bar{p}$ **v q**
Premise 2	0	-1	1		$\bar{q}$ **v r**
Conclusion =	-1	0	1		$\bar{p}$ **v r**

Note the simplicity of the given representation or perspective on the hypothetical syllogism in Figure 11, matched only by the simplicity of the **pqr** string –1, 1, 0, 0, 1-, 1, -1, 0 1, which is merely a set of instructions for displacements in a 3-space.

The vector system can be used in the CNS-space to display other principles, for example implications, by which $\bar{p}$ **v q** implies **p→q**. It also shows that $\bar{p}$ **v q** implies **0 →.** $\bar{p}$ **v q**, as well as $\bar{q} \rightarrow \bar{p}$ and $\bar{p}$ **v** $\bar{q}$ **→0**. Furthermore, the vector system shows nicely the principle of material equivalence, which states that **(p→q)(q→p)** is equivalent to **p↔q** (Figure 13).

The starting point of all these vectors, together with the direction, gives the end point. "Together with" here means treating the points and directions algebraically as themselves directions from the origin. This yields a cancellation technique in which a starting-point of **0** is cancellation of no literal, and an end point of **0** is the cancellation of all the literals

Starting Point	Direction	End Point
0	$\bar{p}$ **v q**	$\bar{p}$ **v q**
p	$\bar{p}$ **v q**	**q**
$\bar{q}$	$\bar{p}$ **v q**	$\bar{p}$
p v $\bar{q}$	$\bar{p}$ **v q**	**0**

Parallel principles can be given for the ANS-plane. Here "$\alpha \rightarrow \beta$" means the same as in the CNS-plane; which is $\alpha \supset \beta$, the conditional, but the reason is hard to see, though interesting. Take the proposition **p → q** in the CNS-plane. It is represented by (among others) an arrow from the point **p** to the point **q**. In the ANS-plane we find an arrow from **p** to **q**. Call it **v**. But what does **v** mean? Note that the CNS-plane vector from **p** to **q** is true if **v** is false. For **v** is **p**$\bar{q}$, and the same vector or direction as **p**$\bar{q}$ **→ 0**. If we want the ANS-plane vectors to represent truth, we must read them as the *denials* of the conjunction of the proposition **p** at the base of the arrow with the negation of the proposition **q** at the arrowhead, or **¬(p**$\bar{q}$**)**. Each arrow in the ANS-plane then reliably represents a conditional.

This reveals something further about the all-important **0**, the origin. We have just learned that in the ANS-plane an arrow from **p** to **q** is $p\bar{q}$, to be read however as a negation. So what does $\alpha \rightarrow \mathbf{O}$ mean? **O** is $p\bar{p}$. So the conjunction of α and $-(\mathbf{O})$ of α-(**pp**). But this is $\alpha(\bar{p}\ \mathbf{v}\ \mathbf{p})$, which is equivalent to α.

Similarly, in the CNS-plane, all the arrows which depart from **0** represent an instance of α **v** **ß**, where α is **O**. Take an arrow from **O** to **p v q**. **O** is the tautology $\mathbf{p\ v}\ \bar{p}$. The negation of this $p\bar{p}$, and so the arrow to the point **p v q** is $-(\mathbf{p\ v}\ \bar{p})\ \mathbf{v\ pq}$. But the first disjunct of this is equivalent to $p\bar{p}$, and so it is always false. Hence the alternation is equivalent to the second disjunct, or the assertion **pq**.

If a vector in the ANS-space is directed towards **O**, **O** has the effect of reversing the truth-values of the base propositions. Moving towards **O** from the base **(p,q)** in the CNS-space we get the vector $\bar{p}\ \mathbf{v}\ \bar{q}$. **O** has the effect of putting **p** and **q** through the Sheffer-function "|". The vector moving *away from* **O** in the CNS-space *towards* e.g. **(p,q)** is also the vector from the base **(p,q)** to **O**, and so it is the vector $\bar{\bar{p}}\ \mathbf{v}\ \bar{\bar{q}}$ or **p v q**.

In a dual fashion, if we are moving *towards* **O** in the ANS-space, we get the base values, so that **pq→O** is **pq**. From **O**, a vector to $(\bar{p},\bar{q})$ will thus be **pq**. In the ANS-space **O** has the effect of putting **p** and **q** through the dagger function "↓", by which **p↓q** is $\bar{p}\bar{q}$.[1]

[1]Wittgenstein's operator N in the *Tractatus* could be described as a generalization of ↓ to more than two places, as N(p,q,r), for example, is **pqr**. We could also describe a generalized Sheffer operation for more than two places which trANS-forms a base such as say (p,q,r,s) into **p v q v r v s**. This operation could be called S for "Sheffer".

Part II

The simplification problem is the problem of reducing truth-functional schemata (or, in the system I am describing, systems of vectors in the ANS-space) to their shortest equivalents. A practical method for doing this, in alternational normal form continues, as Quine observes (Quine, 1982, p. 78), to be suprisingly elusive.

In the ANS-space "vector logic" can be applied to the problem in the following way. Take the schema **pq v p**$\bar{q}$, which as well as implying **p** is equivalent to **p**. To simplify it, form the parallelogram from the origin **0**, **pq** and **p**$\bar{q}$ to the resultant or vector sum point. Call it **i**, for "implicant". The vector acting at **i**, which is in this case **pp**, implies **pq v p**$\bar{q}$. So **i** splits up alternationally, into its components, **pq v p**$\bar{q}$, towards the origin.

Next note that **pq** is equivalent to **pqp**, so that the arrowhead at **pq** can be dragged to **pqp**. But **pp** can also be dragged to **p**. Now we have an arrow from **p to pqp**. But this arrow an be translated into a position on top of the arrow from **p**$\bar{q}$ to **pp**. The same procedure yields a double-headed arrow between **pq** and **pp**, and the result can be read as **pq v p**$\bar{q}$ $\leftrightarrow$ **p**.

When an implicant splits up into its alternations towards the origin, if there is a proposition σ (for "simplest equivalent") at the center of the parallelogram formed by **0**, the disjuncts of a two-clause target schema, and **i**, then σ is a shortest equivalent of the target schema. But this only works for pairs of schemata which do have an **i-point**.

The general simplification procedure, in the ANS-space, is as follows.

(1) Represent the alternational normal schema, the target schema **t**, as a set of vectors in the ANS-space. Each clause or disjunct of **t** is a position vector (i.e. one pointing to **O**) with **O** at one corner of a parallelogram made of propositional addresses to the **i**-point at the other . Any two other outside vertices of such a parallelogram are implicants which are among the original clauses of **t.**

(2) Pick any two clauses. If there is a propositional address σ at the midpoint between the component clauses, the vector from **i** to σ, i.e. σ, is the simplification of and can replace the relevant clauses of **t**, as in the case where **t** is **pq v p$\bar{q}$**, **i** is **pp** and σ is **p**.

(3) Generate **i**-implicants until each clause or vector has been used at least once. If a disjunct **d** of **t** cannot be used because it forms no propositional address with any other disjunct, then **d** must appear unmodified in the final schema which is the simplification of **t**.

(4) If an **i**-point exists in **t**, delete the vectors which produce it in favor of the vector from **i** to **O.**

(5) For a clause in a schema which subsumes another clause, e.g. **pqr v pq**, eliminate the subsuming clause, in this case **pqr**, leaving **pq**. Implications arising from subsumption can be written into the whole vector system of **t** as components where relevant. For example, an arrow can be drawn from **pqr** to **pq** in the above example.

Rule (5) applies for example to **pq v p**, which is an undeveloped or unbalanced schema in which **pq** subsumes **p**. How does **pq v p** simplify to **p**, when it seems to yield **p v q**? The Answer, which cashes the metaphor of "subsumption", is that **p** really represents a plane, in a 3-space, sweeping out the whole **p**-domain, or any ANS-schema with **p** in it. So it is a kind of type fallacy to represent **pq** alongside **p** in a single schema as if they were to be treated separately. For **pq**, and **p$\bar{q}$**, are really "elements" of **p** itself. A cube is not so many faces and so many lines, but it can be represented as lines producing faces or vice versa. As a matter of philosophy, therefore, vector logic can avail itself, as rule (5) does, of a preliminary use on Quine's operation (i) from "A Way to Simplify Truth Functions", which has us 'drop the subsuming clause ... if one of the clauses of alternation subsumes another...'. Quine's operation (i) also replaces α **v** $\bar{\alpha}\ \phi$ with α **v** ϕ, and the same for the corresponding α -schemata (Quine, 1955, p. 627).

(6) Couples such as **pq v $\bar{p}\,\bar{q}$** or **$\bar{p}\,\bar{q}$s v pq$\bar{s}$** cannot be summed to zero, the origin.

(7) Translate vectors as in Figure 14. Any superpositions of parallel arrows in opposite directions represent equivalences. (a) Drop the longer clause at the end of any double-headed arrow. (b) Drop pairs, triples etc. of double-headed arrows which meet at a point in favor of the vector from that point to **0**. (c) Drop a vector or clause in the target schema which is itself the resultant of any other two vectors.

(8) A simplification is complete if in the system which replaces the target schema: (a) no vectors or clauses are subsumed by others (see Rule 5); (b) no double-headed vectors remain, or, in other words, if all equivalences in the system have been exploited.

Take next the simplification of the four-clause target schema $\mathbf{pqr} \vee \mathbf{pq}\bar{r} \vee \bar{p}\mathbf{qr} \vee \mathbf{p}\bar{q}\bar{r}$. The first job is to plot the target schema in a V-diagram. We get two parallelograms, with two **i**-points, **qrqr** and $\mathbf{p}\bar{r}\mathbf{p}\bar{r}$, and two σ-points, **qr** and $\mathbf{p}\bar{r}$, which are final in the sense that they do not generate a further σ-point. Hence the target schema is equivalent to $\mathbf{qr} \vee \mathbf{p}\bar{r}$.

Now take the simple-looking three-clause schema $\mathbf{pqr} \vee \mathbf{pq}\bar{r} \vee \mathbf{p}\bar{q}\bar{r}$. The resultant is $\mathbf{pq} \vee \mathbf{p}\bar{r}$.

Here the σ- and **i**- points function as before. But something else has happened. The vector $\mathbf{pq}\bar{r}$ has been used twice, once along with **pqr** to give **pq**, and again, with $\mathbf{p}\bar{q}\bar{r}$, to give $\mathbf{p}\bar{r}$. Why was $\mathbf{pq}\bar{r}$ not exhausted by its first use, and why can it be used again? The Answer can be seen by looking at the truth-table for $\mathbf{pqr} \vee \mathbf{pq}\bar{r} \vee \mathbf{p}\bar{q}\bar{r}$, which is

1. **pqr**	**T**
2. $\mathbf{pq}\bar{r}$	**T**
3. $\mathbf{p}\bar{q}\mathbf{r}$	
4. $\mathbf{p}\bar{q}\bar{r}$	T
5. $\bar{p}\mathbf{qr}$	

6. $\bar{p} q \bar{r}$

7. $\bar{p} \bar{q} r$

8. $\bar{p} \overline{qr}$

Truth, it could be said, is not exhausted by use. The $\mathbf{p}\bar{r}$ of line 2 is so to speak redundant, as line 2 has already been captured by the disjunct **pq**, and so line 4 has had half of its work already done.

This simplification procedure is theoretically an improvement on the techniques used in Karnaugh maps (Garrod and Borns, 1991, p. 153 ff.), as it needs no wrapping around and can be used mechanically and easily on more than four variables – any number fits into the "proposition circuit", which gradually turns from a square, with two variables, into a hexagon, with three, and finally into a circle, with an infinite number of variables. With four variables, the logical space is as given in Figure 17.

The whole figure in Figure 17 is a "measure polytope" or hypercube, though one with a further complex internal structure. There is no limitation of tessellation to the number of propositional variables or vectors **p, q, r s** ... that can be handled, because the space is derived not from a closed figure, such as a cube, but from a sheaf of lines in the geometrical sense. Not all closed figures tessellate. All the lines of the multi-dimensional sheaves are coincident.

Consider in Figure 17 a simplification from $pqr\bar{s}$ v $pq\bar{r}\bar{s}$ v $pq\bar{r}s$ v $\bar{p}qr\bar{s}$ v $p\bar{q}\bar{r}s$ v $\bar{p}q\bar{r}\bar{s}$ to $p\bar{r}s$ v $q\bar{s}$. This proceeds as shown, with the six vectors reducing to two. The first vector or σ-point in the simplification, $p\bar{r}s$, results from the implicand pair $pq\bar{r}s$ v $p\overline{qr}s$, dropping the up-down $q/\bar{q}$ component. In this case we have a simplification from two four-letter schemata to one three-letter schema. The remaining four schemata are all needed to fix the σ-point $q\bar{s}$. Both pairs $pqr\bar{s}$ v $p\bar{q}\bar{r}s$ and $pq\bar{r}\bar{s}$ v $\bar{p}qr\bar{s}$ must give a fix on the same σ-point if the reduction from four literals to two is to be justified. For either pair by themselves is not sufficient for the required biconditional. The general rule is

$$\mathbf{V=2d}$$

where **v** is the number of vectors required to make the fix on the σ-point, and **d** is the drop in the number of literals from the clauses of the given schema to the resulting clause in the target schema.

Another illustration of the Fix Rule is pqr v $pq\bar{r}$ v $p\bar{q}r$ v $p\bar{q}\bar{r}$ v $\bar{p}qr$ v $\bar{p}q\bar{r}$ v $\bar{p}\bar{q}r$, which is , however, equivalent merely to **p v q v r**, as it covers every line of the truth-table except $\overline{pqr}$.

One pair of implicands is $p\bar{q}r$ v $pq\bar{r}$, which give the σ-point **p**. But as this is a drop down from three letters to one, we need a fix of four vectors or two vector sums on the point, and the third and fourth vectors $\bar{p}qr$ and $pq\bar{r}$ provide it. The same sort of fix appears with **q** (pqr v $\bar{p}q\bar{r}$ and $\bar{p}qr$ v $pq\bar{r}$) and **r** ($\bar{p}\bar{q}r$ v pqr and $\bar{p}qr$ v $p\bar{q}r$).

A much simpler though negative example of the Fix Rule is $\mathbf{pq}\bar{r}$ **v** $\mathbf{p}\bar{q}\bar{r}$, which seems to give **p** as an σ-point resultant, but fails to for lack of a fix on the point **p**, as four, not two vectors must converge on it for the drop. This acts as a constraint on the vector arithmetic. We seem to get

	p	q	r	
	1	1	1	+
	1	-1	-1	
=	1	0	0 .	

But the Fix Rule rules this out. If x columns are filled with numbers, positive or negative, then the number of non-zero columns in the sum must be **x-1**. The Fix Rule will seem entirely unartificial when one recognizes that what it means in, say, a 3-space, is that a literal or one-letter proposition is a *face*, and so four corners are needed to determine it. A two-letter proposition is a *line*, and so only two letters are needed to fix it. And a point in a 3-space is a three-letter proposition.

Consider as another illustration of the Fix Rule **pqrs v** $\mathbf{pqr}\bar{s}$ **v** $\mathbf{pq}\bar{r}\mathbf{s}$ **v** $\mathbf{pq}\bar{r}\bar{s}$ **v** $\mathbf{p}\bar{q}\mathbf{rs}$ **v** $\mathbf{p}\bar{q}\mathbf{r}\bar{s}$ **v** $\mathbf{p}\bar{q}\bar{r}\mathbf{s}$ **v** $\mathbf{p}\bar{q}\bar{r}\bar{s}$. This is very obviously equivalent to **p**, and since d = 3 for each clause, **v** for the point **p** = 2d. So **d** = 8, and eight vectors or four vector sums are needed for the fix on the σ-point.

In many cases the target schema is unbalanced in the sense that its clauses have different numbers of conjuncts and so they need to be put into developed alternational normal form. An example **pq v** $\mathbf{p}\bar{q}\mathbf{r}$ **v** $\bar{p}\bar{q}\bar{r}$ (Quine, 1982, p. 75). This is equivalent to **pq v pr v** $\bar{p}\bar{q}\bar{r}$. Like the early Quine's procedure in "The Problem of Simplifying Truth Functions" (Quine, 1952, p. 524), the vector simplification method given so far has taken the cumbersome 'developed normal formulas as the point of departure.'

If **t** is developed uniformly we get $\bar{p}\bar{q}\bar{r} \vee p\bar{q}r \vee pqr \vee pq\bar{r}r$, which in a **V**-diagram is clearly $\bar{p}\bar{q}\bar{r} \vee pq \vee pr$, as **pr** lies midway between **pqr** and **0**, and **pq** lies midway between **pqr** and $pq\bar{r}$. But without development, we can take the iota-point for $p\bar{q}r \vee pq$, which is **pr**, and argue that since $pr \rightarrow .p\bar{q}r \vee pq$ (where ".expression" and "expression." represents bracketing of the "expression" that precedes or follows the dots), and **pr** subsumes $p\bar{q}r$, the longer $p\bar{q}r$ can simply be replaced by its own implicant.

The equivalence of undeveloped

(i) $\bar{p}q \vee p\bar{q} \vee \bar{q}r \vee q\bar{r}$

and

(ii) $p\bar{q} \vee \bar{p}r \vee q\bar{r}$

is harder to establish. It is one which resists as many as twelve fell swoops (Quine, 1982, p. 76, also in 1952, pp. 523-527) or shorter truth-tables. In the vector space with developed alternational forms the equivalence is easy enough to see. The developed form of this equivalence is easy enough to see. The developed form of this example is: $\bar{p}qr \vee \bar{p}\bar{q}r \vee \bar{p}q\bar{r} \vee pq\bar{r} \vee p\bar{q}\bar{r} \vee p\bar{q}r$.

The same result can be obtained using column matrices for the pairs of vectors. Then for $\bar{p}qr \vee \bar{p}\bar{q}r$ we get

p	q	r		
-1	1	1	+	$\bar{p}qr$
-1	-1	1		$\bar{p}\bar{q}r$
-1	0	1		$\bar{p}r$

And for $\bar{p}q\bar{r}$ v $pq\bar{r}$ we get

	p	q	r		
	-1	1	-1	+	$\bar{p}q\bar{r}$
	1	1	-1		$pq\bar{r}$
=	0	1	-1		$q\bar{r}$

Similarly, for $p\bar{q}r$ v $p\bar{q}\bar{r}$ we get

	p	q	r		
	1	-1	1	+	$p\bar{q}r$
	1	-1	-1		$p\bar{q}\bar{r}$
=	1	-1	0		$p\bar{q}$

It should be noted that in this example too the Fix Rule applies. It would be nice to take the vectors in a different order, so that $p\bar{q}r$ and $pq\bar{r}$ are chosen instead of $p\bar{q}r$ and $p\bar{q}\bar{r}$, and also $\bar{p}q\bar{r}$ and $p\bar{q}\bar{r}$ instead of $\bar{p}q\bar{r}$ and $pq\bar{r}$. This would yield **r** and **p** instead of **pq** and $q\bar{r}$ in the whole system. But that would mean dropping from three letters to one in the case of these two pairs of alternations, and we cannot do that as there is no fix on **r** or on **p**.

There may of course be more than one "shortest" schema. In Quine's example there is obviously is on inspection a second. The vector system $\bar{q}r$ v $\bar{p}q$ v $p\bar{r}$ has the same overall "effect" in the vector-logical space.

Let us now try this example with the use of the **i**-points, the key prime implicants. Take first $\bar{p}q$ v $\bar{q}r$. This alternation is implied by the **i**-vector which forms the parallelogram with **0**. But there is no Φ-point, and so, apparently, $\bar{p}q$ v $\bar{q}r$ is not

equivalent to $\bar{p}\,\mathbf{r}$. Yet in the context of the whole scheme $\mathbf{p}\bar{q}$ v $\bar{q}\,\mathbf{r}$ v $\bar{p}\,\mathbf{q}$ v $\mathbf{q}\bar{r}$, it is. To see this, we move the free vectors $\mathbf{p}\bar{q}$ and $\mathbf{q}\bar{r}$ from the right-hand side of the **V**-diagram to the parallelogram on the left. The implicand of $\mathbf{p}\bar{r}$, which is $\mathbf{p}\bar{q}$ v $\mathbf{q}\bar{r}$, slides into place from $\mathbf{p}\bar{q}$ v $\mathbf{q}\bar{r}$ back to $\bar{p}\,\mathbf{r}$, and the two-way implication or equivalence is established.

The vector summation of $\mathbf{p}\bar{q}$ and $\bar{q}\,\mathbf{r}$ to $\mathbf{p}\overline{qq}\mathbf{r}$ or $\mathbf{p}\bar{q}\,\mathbf{r}$ is disallowed by the Fix Rule, according to which the number of vectors needed to make a fix is equal to the d-th power of 2. This summation would actually produce a negative value for d. As the number of literals rises from two to three, the drop *increases* from 2 to 3, or –1.

Quine gives another interesting example of a simplification with four simplest equivalents, one which also illustrates the method of simplification for non-developed or unbalanced schemata like the last example. The example (Quine, 1952, p. 528) is **pqr v** $\mathbf{p}\bar{r}$ v $\mathbf{pq}\bar{s}$ v $\bar{p}\,\mathbf{r}$ v $\bar{p}\,\bar{q}\,\bar{r}\,\bar{s}$ (Figure 23).

We begin by generating vector sums for the various disjuncts. We can see fairly easily that $\bar{p}\,\mathbf{r}$ and $\bar{p}\,\bar{q}\,\bar{r}\,\bar{s}$ to start with, yield a parallelogram, but it seems to end at an **i**-point outside the logical space. Yet if we study that point, we can see that it is actually at the co-ordinates $\bar{p}\,\bar{q}\,\bar{r}\,\bar{s}\,\bar{p}\,\mathbf{r}$. This point, however, contains a contradictory or backward *and* forward instruction, namely the **r** from $\bar{p}\,\mathbf{r}$ and the $\bar{r}$ from $\bar{p}\,\bar{q}\,\bar{r}\,\bar{s}$ which can both be deleted. There is also a double **p** in the final address, and one **p** of these, but not both, can of course be deleted. This leaves an end-point for a vector $\bar{p}\,\bar{q}\,\bar{s}$. By similar reasoning, we can arrive at the vector $\bar{q}\,\bar{r}\,\bar{s}$ as the vector sum of $\mathbf{p}\bar{r}$ and $\bar{p}\,\bar{q}\,\bar{r}\,\bar{s}$. And similarly **pqr** with $\mathbf{p}\bar{r}$ gives **pq**, or **pqr** with $\bar{p}\,\mathbf{r}$ gives the **i**-point **qr**. Each vector must be used at least once if it is not to appear unchanged in the simplified schema.

The corresponding Φ-points, however, do not appear at the designated addresses which are shortened versions of their **i**-points, and so the equivalence of the **i**-points and

their implicands is not established. Just as in the example shown in Figure 20, the drag-back effects described in connection with **p** and **pp** in Figure 14 do not apply.

As before, in Figure 24, we first write in the parallelogram from **0** for the pair **pqr** v **p**$\bar{r}$. This gives an Φ-point at **pqp**, and so from **pqp** we write in a pair of vectors to **pqr** and **p**$\bar{r}$. Now **pqr** implies **pq**, and is subsumed by it, and we can represent the subsumption rule here by drawing in the vector from **pqr** to **pqp**. Figure 24 is now showing an *equivalence* between **pqr** and **pq**, but *only in the presence of the* $p\bar{r}$ *in the alternational schema* ***pqr*** ***v*** $p\bar{r}$, i.e. with the translated or "borrowed" vector **p**$\bar{r}$ → **0**.

It is worth realizing that in the reductions in developed normal form, the implicands can be replaced by the implicant only because of the various biconditionals or double arrows at work. There is no intrinsic magic in the Φ-point. In the undeveloped examples, too, clauses do not disappear in a general way because pairs of disjuncts collapse into their implicants, but because of the presence of specific conditions elsewhere in the schema, which translated have the effect or creating biconditionals.

Finally, why is **pq**$\bar{s}$ superfluous in the example given in Figure 23? The Answer is interesting and complicated, and principles about superfluity need to be established.

Take the truth, sometimes known as the Consensus Theorem, that **pq v p**$\bar{r}$ **v qr.** **↔.pq v** $\bar{p}$ **r.** Representing this in the ANS-space for **p**, **q** and **r**, we can see that the implicant **qr** is the resultant of the disjunction of **pq** and $\bar{p}$ **r** (Figure 25). We can give it as a general truth that implicants, in the ANS-space, are resultants.

Let the left-hand side of the Consensus Theorem be represented as

qp v $\bar{p}$ **r v qr**

The Theorem says that the disjunct **qr** is superfluous. Consider the dual of the left-hand side of the Theorem, in the CNS-space. It is

$$(q \vee p)(\bar{p} \vee r)(q \vee r)$$

This is the conjunction ($\bar{q}$ **p)(p r)**($\bar{q}$ **r).** But clearly the last conjunct is superfluous, as the first two conjuncts imply it by a hypothetical syllogism, in the sense that if they are true, so is it (Figure 25).

It is nice to see the dual roles of conjunction and alternation, or ANS-and CNS-spaces, truth and falsehood, and how the concept of the *resultant* and the *component* binds them together.

In the following truth-table, we can that the **(q r)** resultant is so to speak "covered" by its component with respect to truth, in the ANS-space, and falsity in the CNS-space. That is, with the disjunctions in the ANS-space the addition of an extra truth on already true lines of the truth-table does not affect the truth of the whole schema. And similarly in the CNS-space, if the whole schema is already false, adding a false conjunct will not affect that result.

					ANS-				CNS-	
p	q	r	**qr**	**v**	**pq**	**v**	$\bar{p}$**r**	**(q v p)**	**(p v r)**	**(q v r)**
T	T	T	T		T				F	
T	T	F			T					
T	F	T								
T	F	F							F	F
F	T	T	T				T			
F	T	F								
F	F	T					T	F		
F	F	F						F		F

We are now in a position to deal with the superfluity of **pq**$\bar{s}$ in Quine's example in Figure 24. Disjunctive clauses in the ANS-space like **pq** $\bar{s}$ are superfluous when they are components. Before the representation of the schema **pqr v** $\bar{p}\ \bar{q}\ \bar{r}\ \bar{s}$ **v p**$\bar{r}$ **v** $\bar{p}$ **r v pq**$\bar{s}$ with a view to simplification, we can simply run a check to see if any of the clauses are implicants or iota-points for any others. We can easily find that **pq**$\bar{s}$ **→ .pqr v p**$\bar{r}$ from the truth-table for the schema; all the lines on which **pq**$\bar{s}$ is true are also lines on which either **pqr** or **p**$\bar{r}$ is already true, and so **pq**$\bar{s}$ can be deleted from the schema to be simplified.

Geometrically, the construction is as follows: (Figure 27). Note that **pq**$\bar{s}$ extends to **pq**$\bar{s}$**p**. This is however the implicant for **pqr**$\bar{s}$ **v p**$\bar{r}$. However, **pqr**$\bar{s}$ itself extends to **pqr**$\bar{s}$**pq**. This last schema is the implicant for **pqr v pq**$\bar{s}$. So **pq**$\bar{s}$ gives way to **p**$\bar{r}$ **v pqr**$\bar{s}$. But **pqr**$\bar{s}$ can itself can be dropped in favor of **pqr v pqr**$\bar{s}$. Any line of the

truth-table for **pqr** on which is true is also one which either **pqr** or **p**$\bar{r}$ is already true. Hence **pq**$\bar{s}$ can be dropped.

So for Quine's example in Figure 23 we are left with the four possibilities:

These examples, and others like them, suggest the possibility of further applications of simplifying geometrical theorems and methods to the simplification problem.

The charm of a vector simplification technique is that is follows a least-action principle, for any number of propositional vectors, in the sense that the problem is not one of finding shortest equivalents to truth-functional schemata. Rather the space, inasmuch as it is fixed vector space in which all free vectors having the same direction are in a sense the *same* directional vector, is unable *not* to give the desired result.

As to propositional logic as whole, it is nice to have all of the nineteen or however many clanking "rules of inference" within the space, so that there is just the one intuitively obvious method of argument: vector addition. It is really absurd to think of empty or "formal" rules such as association and communication as having the same status as say modus tollens, which is genuine "motor" that advances arguments through logical space. Association and commutation should flow out of the nature of the logical space, and in the vector space they do. The vectors **p v q** and **q v p**, for example, have the same end-point, though they arrive at it by different but corresponding routes.

Part III

(i) Electrical and Integrated Circuit Minimization

Let us now see how the techniques described can be used in a routine for simplifying electrical and integrated circuits. Take the target circuit **ABC** + **A**$\overline{C}$ + **AB**$\overline{D}$ + $\overline{A}$**C** + $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ (Figure 28).

The first job is to plot this in the ANS-space as the set of vectors **pqr v p**$\bar{r}$ **v pq**$\bar{s}$ **v** $\bar{p}$**r v** $\bar{p}\,\bar{q}\,\bar{r}\,\bar{s}$, as in Figure 23 above. Following the routine (1)-(8) given on p. 20, above, we can simplify this system of vectors to e.g., **pq v** $\bar{p}$**r v pr v** $\bar{p}\,\bar{q}\,\bar{s}$ (cf. p. 39).

The resultant schema can then be translated into the circuit diagram $\mathbf{AB} + \mathbf{A}\bar{C} + \bar{A}\mathbf{C} + \bar{A}\,\bar{B}\,\bar{D}$ (Figure 29).

Note that the target circuit has five gates (G=5), that the total number of inputs into these gates is twelve (I=12), and that the redundancy factor (i.e., the number of times an original input is used again, corresponding to the join dots) is seven (R=7). These figures drop to G=4, I=9 and, most importantly, R=2 for the simpler circuit, representing corresponding gains in materials savings, speed and reliability.

(ii) Free Space Optical Computation

More than ten years ago the National Academy of Sciences Panel on Photonics Science and Technology Assessment declared that 'The ultimate benefit of photonic processing could occur if practical optical logic could be developed' (Whinnery et. al., *Photonics*, 1988, p. 35). So far the implied challenge of the Panel has not been met.

Vector manipulation has been one of the big success stories for optical computation, but vector techniques themselves promise an application to the logic of optical computation as a whole. The full ANS-/CNS- space could be built as an optical device for checking the validity of arguments or as a logic device for optical computation, and also as simplification machine. Each operation in the space is a laser, and the resultant proposition-points such as **p** and **pq** and **pqr** are multifaceted beamsplitters or mirrors which reflect the beams in the correct logical directions at the correct logical strengths to ensure the required implications.

Thus in Figure 30 a beam **V** can be sent from the origin to a half-darkening beamsplitting mirror at the node p. At p it is split and sent at half-strength to q, and to $\bar{q}$. Simultaneously, a second beam **U** from **O** is sent to the node $\bar{p}$ **v q**, which is also **p→q.** At this point **U** is split and sent at half-strength to $\bar{p}$ and to **q**. The proposition **p** is said to "half-imply" **q**, in the sense that with one other proposition it *does* imply **q**, and the proposition $\bar{p}$ **v q** is said to "half-imply" **q** in the sense that with one other proposition (**p**) it *does* imply **q**.

Both half-implication beams are coincident on **q**, and at **q** the photoreceptor gives a reading of .5 + .5 or 1. The system has optically computed *modus ponens*; from an

input of **p** and an input of $\overline{p}$ **v q**, it has yielded up **q**. The system gives a physical interpretation of beamsplitting as multiple implication and of darkening as fractional implication.

The same principles will apply to the other rules of inference and logical equivalences.

A development of the system given for *modus ponens* in Figure **30** obviates the need for a free beam for e.g., **p** to **q**, and simplifies the design of the node. In Figure **31,** the beam to **p** is split, at full-strength, to **p**'s implicants, which are **p v q** and $\overline{p}$ **v q** (ignoring tautologies). The beamsplitter at $\overline{p}$ v q itself directs the beam to **q** at only half-strength, and the desired computation is achieved.

We can also arrange that in an embodiment of the uninterpreted **(p, q, … n)** space, in which the base **(p,q)** is either **p v q** or **pq** (though not both), configurations of the beamsplitters will allow the node to switch between the two states. A conjunctional state will correspond to a *concave* configuration, as both inputs are required for the activation of the node, and an alternational state will correspond to a *convex* configuration, as shown in Figure 32.

(iii) "Flat" Optical Processing

A second method of exploiting the vector system for computation is more markedly spatial. Represent the propositions in the uninterpreted **(p,q)** space with spatial light modifiers (SLMs). When the first premise is input, e.g. $\overline{p}$ **v q,** then the origin **0**, and with it the position of the whole space, are moved to the point $\overline{p}$ v q, or in a $\overline{p}$ **v q** direction. We could say the **0** *becomes* $\overline{p}$ **v q**, so that we are now in a $\overline{p}$ **v q** environment, a $\overline{p}$ **v q** world. Then **p** in the second SLM (Figure 33) will *be* **q**, and we have modus ponens. And when the whole space is displaced in a $\overline{p}$ **v q** direction, **q** *is* **p**! The SLMs are shown sequentially in Figure 34.

(iv) Colorimetric Processing

Colored laser beams can be used so that the refractive angle is built into the vector rather than into the propositional nodes as the CIE (Commission Internationale de l' Eclairage) x-y chromaticity diagram (a color mixing diagram) is itself a vector space. (Or a mixed

system of colored laser and colored mirrors could be used.) Then optical computation for simplification is simply the colorimetric process of additive color mixing. IN the CNS-space let **p** be red (R), $\bar{p}$ a complementary cyan blue-green (C), **q** a yellow (Y), $\bar{q}$ a complementary blue (B), **p v q** yellow-red (YR) and $\bar{p}$ **v q** blue-red (BR). Also $\bar{p}$ **v** $\bar{q}$ is the complementary of YR, a cyan blue.

The contradiction of **0** (the so-called "Nullpunkt", or "white") corresponds to the addition of complementary hues.[2] **YR + BR = R**, since **Y** and **B** are complementary.

With these colorimetric assignments we can compute modus ponens and the other rules of argument and truth-preserving substitutions. The proposition **p** is **R**, and **q** is **Y**. So **p→q**, or $\bar{p}$ **v q**, is **CY**. Together with **R** this give **Y** or **q**, as **C** and **R** are complementaries.

In the ANS- space we can perform simplifications colorimetrically. Take the most basic simplification as an illustration, in which **pq v p**$\bar{\mathbf{q}}$ is equivalent to **p**. Let **YR** represent **pq** and **BR p**$\bar{\mathbf{q}}$. The **Y** and **B** beams cancel to the "Nullpunkt", leaving **RR** or **R**, which is **pp** or **p** (Figure 36)

It is still true, as Norbert Streibl et. al. ("Digital Optics", (1989)) pointed out, the 'A uniform technology for digital optical information processing, comparable in its significance to microelectronics, does not yet exist and is by itself a challenging research goal.' A vector logic for optics is a source from which such a "uniform" technology can flow, just as electronics derived from the natural isomorphism of electric circuitry and truth-functional logic.

In this disclosure, there is shown and described only the preferred embodiment of the invention, but, as aforementioned, it is to be understood that the invention is capable of use in various other combinations and environments and is capable of changes or modifications within the scope of the inventive concept as expressed herein.

[2] With complementaries '… what is offered, so to speak, in the way of colour by one spectrum (or colour) is withdrawn by the other, so that the result is a vanishing of colour, just as in a contradiction between two propositions which negate one another the result is a vanishing of information' (Jonathan Westphal, *Colour,* Blackwell, 1991, p. 108.

Claims:

1. A method of designing logical circuits, comprising the steps of:
   a. representing the logic of a logical circuit to be designed as points and vectors in a vector space; and
   b. using the points and vectors in a vector space to simplify the logic of the logical circuit to a simpler form; and
   c. designing the logical circuit using the simpler form.

2. A method of manufacturing logical circuits, comprising the steps of:
   a. representing the logic of a logical circuit to be manufactured as points and vectors in a vector space; and
   b. using the points and vectors in a vector space to simplify the logic of the logical circuit to a simpler form; and
   c. using the simpler form to implement the logical circuit in hardware.

3. A method of simplifying logical circuits, comprising the steps of:
   a. representing the logic of a logical circuit as points and vectors in a vector space; and
   b. modifying the representation in vector space using at least one process rule of a set of process rules to simplify the logic.

4. The method of claim 3 in which at least one process rule of a set of process rules consisting of the following process rules:
   a. Process Rule 1--

      a1. Represent the alternational normal schema, the target schema **t**, as a set of vectors in the ANS-space,

      a2. Each clause or disjunct of **t** is a position vector (i.e. one pointing to **O**) with **O** at one corner of a set of parallelograms made of propositional addresses to the **ℹ**-point at the other,

      a3. Any two other outside vertices of such a parallelogram are implicants which are among the original clauses of **t**;

   b. Process Rule 2--

      b1. Pick any two clauses,

      b2. If there is a propositional address σ at the midpoint between the component clauses, the vector from **ℹ** to σ, i.e. σ, is the simplification of and can replace the relevant clauses of **t**, as in the case where **t** is **pq v p$\bar{q}$**, **ℹ** is **pp** and σ is **p**;

   c. Process Rule 3--

      c1. Generate **ℹ**-implicants until each clause or vector has been used at least once,

c2. If a disjunct **d** of **t** cannot be used because it forms no propositional address with any other disjunct, then **d** must appear unmodified in the final schema which is the simplification of **t**;

d. Process Rule 4--

d1. If an **i**-point exists in **t**, delete the vectors which produce it in favor of the vector from **i** to **O**;

e. Process Rule 5--

e1. For a clause in a schema which subsumes another clause eliminate the subsuming clause;

f. Process Rule 6--

f1. Couples such as $\mathbf{pq} \vee \bar{p}\bar{q}$ or $\bar{p}\bar{q}s \vee \mathbf{pq}\bar{s}$ cannot be summed to zero; the origin.

g. Process Rule 7--

g1. Translate vectors as in Figure 14 if a corresponding σ-point exist for a **i-point** then 6 is the simplification of **i**.

g2. Any superpositions of parallel arrows in opposite directions represent equivalences,

g3. For equivalences, (a) Drop the longer clause at either end of any double-headed arrow, **(b)** Drop pairs, triples etc. of double-headed arrows which meet at a point in favor of the vector from that point to **O** and (c) Drop a vector or clause in the target schema which is itself the resultant of any other two vectors;

h. Process rule 8--

h1. A simplification is complete if in the system which replaces the target schema no vectors or clauses are subsumed by others and no double-headed vectors remain (i.e. if all equivalences in the system have been exploited).

5. Apparatus for simplifying logical circuits, comprising:
   a. a processing element configured to represent the logic of a logical circuit to be simplified as points and vectors in a vector space and to use the points and vectors to simplify the logic of the logical circuit to a simpler form.

6. The apparatus of claim 5 in which the processing element is an optical computer.
7. The apparatus of claim 5 in which the processing element is a digital computer.

8. The apparatus of claim 1 in which the processing element is an colorimetric computer.
9. The apparatus of claim 1 in which the processing element is an analog computer.
10. A computer program product, comprising:
    a. a memory element; and
    b. a computer program stored on said memory medium, said computer program comprising instructions for representing the logic of a logical circuit to be designed as points and vectors in a vector space and for using the points and vectors in a vector space to simplify the logic of the logical circuit to a simpler form and for designing the logical circuit using the simpler form.

11. A computer program product, comprising:
    a. a memory element; and
    b. a computer program stored on said memory medium, said computer program comprising instructions for representing the logic of a logical circuit to be manufactured as points and vectors in a vector space, and for using the points and vectors in a vector space to simplify the logic of the logical circuit to a simpler form, and for using the simpler form to implement the logical circuit in hardware.

12. A computer program product, comprising:
    a. a memory element; and
    b. a computer program stored on said memory medium, said computer program comprising instructions for representing the logic of a logical circuit as points and vectors in a vector space, and for modifying the representation in a vector space using at least one process rule of a set of process rules to simplify the logic.

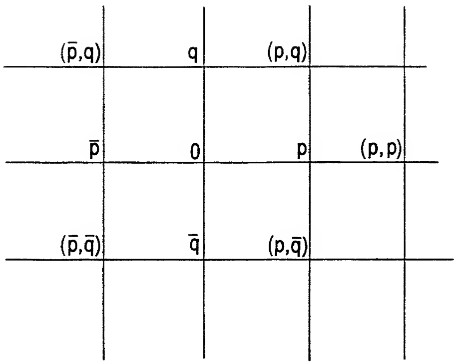

Figure 1
A Space for Propositions

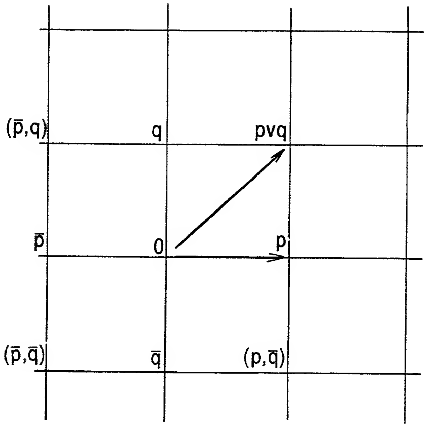

Figure 2

The Propositions p and q v q

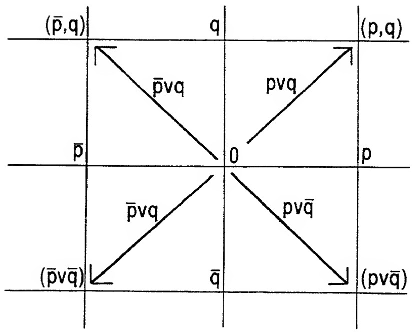

Figure 3
Two-Letter Alternational Clauses

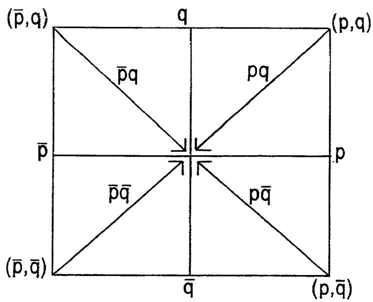

Figure 4
The ANS-plane

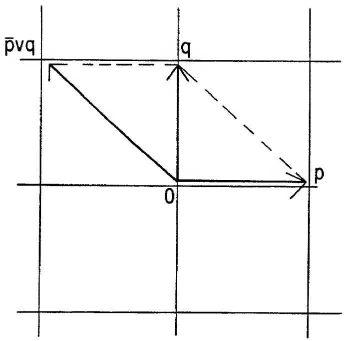

Figure 5
Modus Ponens in the CNS-plane

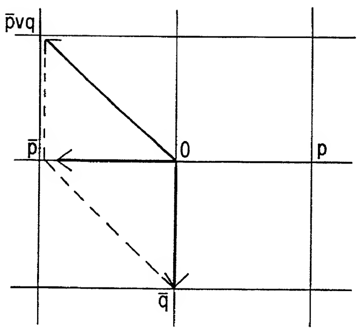

Figure 6
Modus Tollens in the CNS-plane

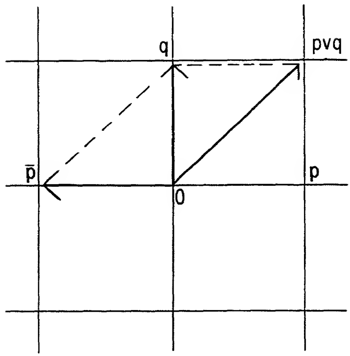

Figure 7
The Disjunctive Syllogism in the CNS-plane

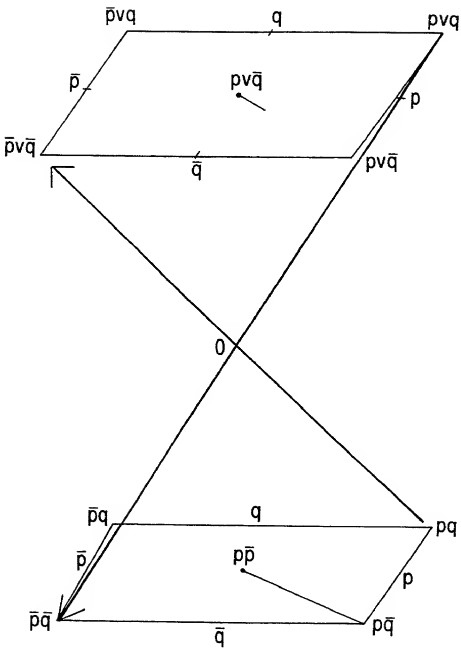

Figure 8
Harmony of ANS-and CNS-planes

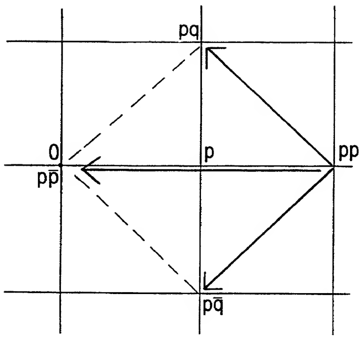

Figure 9
pp +, pq v pq

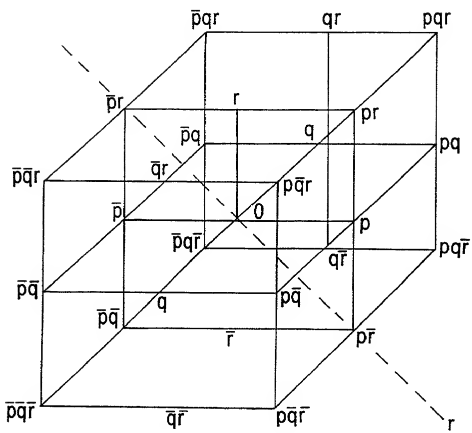

Figure 10

2D Representation of the 3D ANS-Space

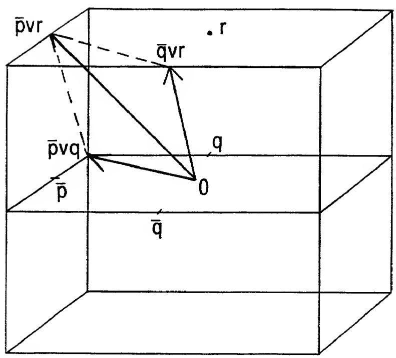

Figure 11

The CNS-space with Three Variables: the Hypothetical Syllogism

	p	q	r		
Premise 1	-1	1	0	+	$\bar{p}$vq
Premise 2	0	-1	0		$\bar{q}$vr
Conclusion =	-1	0	1		$\bar{p}$vr

Figure 12A

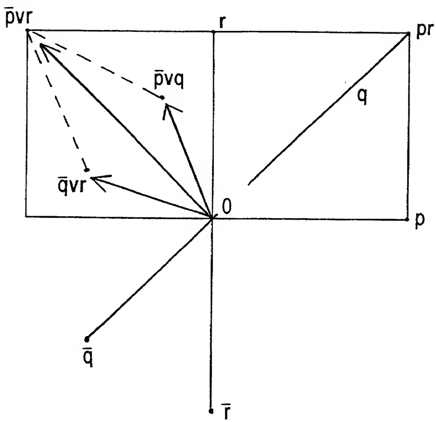

Figure 12B

Hypothetical Syllogism in a 3D CNS-Space

Starting Point	Direction	End Point
0	$\bar{p} \vee q$	$\bar{p} \vee q$
$\cancel{p}$	$\cancel{\bar{p}} \vee q$	q
$\cancel{\bar{q}}$	$\bar{p} \vee \cancel{q}$	$\bar{p}$
$\cancel{p} \vee \cancel{\bar{q}}$	$\cancel{\bar{p}} \vee \cancel{q}$	0

Figure 13A

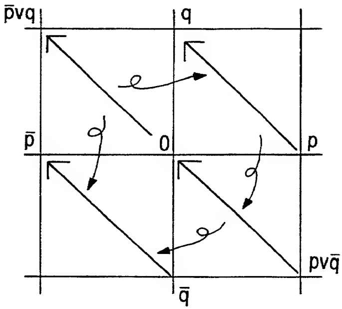

Figure 13B

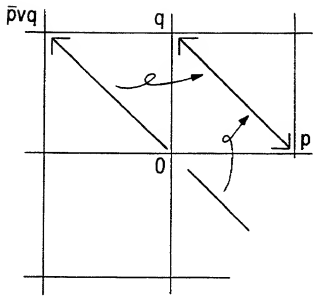

Figure 13C

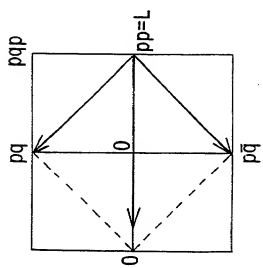

Figure 14A

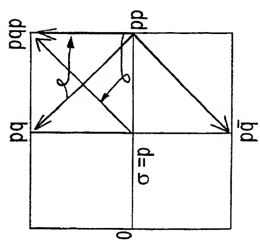

Figure 14B

$pq \vee p\bar{q} \leftrightarrow p$

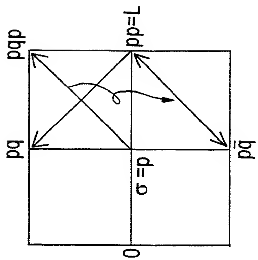

Figure 14C

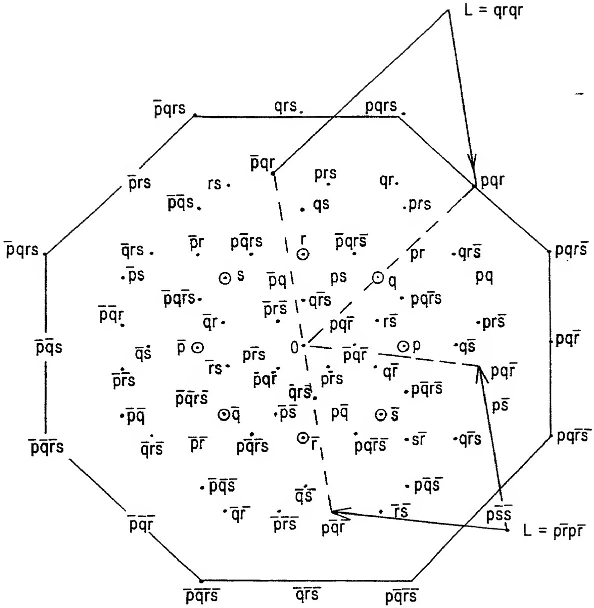

Figure 15

A Four-Clause Schema Simplified

$pqr \vee pq\bar{r} \vee \bar{p}qr \vee p\bar{q}\bar{r}. \longleftrightarrow .qr \vee p\bar{r}$

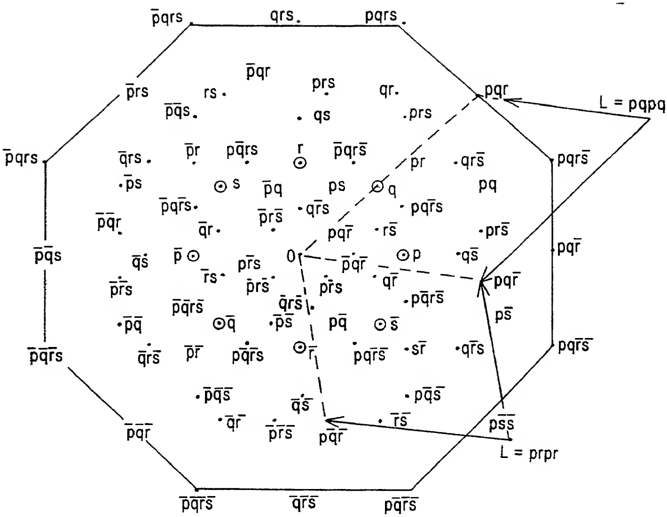

Figure 16A

$pqr \vee pq\bar{r} \vee p\bar{q}\bar{r} \vee p\bar{q}\bar{r}. \longleftrightarrow .qr \vee p\bar{r}$

Figure 16B

1.	pqr	T
2.	$pq\bar{r}$	T
3.	$p\bar{q}r$	
4.	$p\bar{q}\bar{r}$	T
5.	$\bar{p}qr$	
6.	$\bar{p}q\bar{r}$	
7.	$\bar{p}\bar{q}r$	
8.	$\bar{p}\bar{q}\bar{r}$	

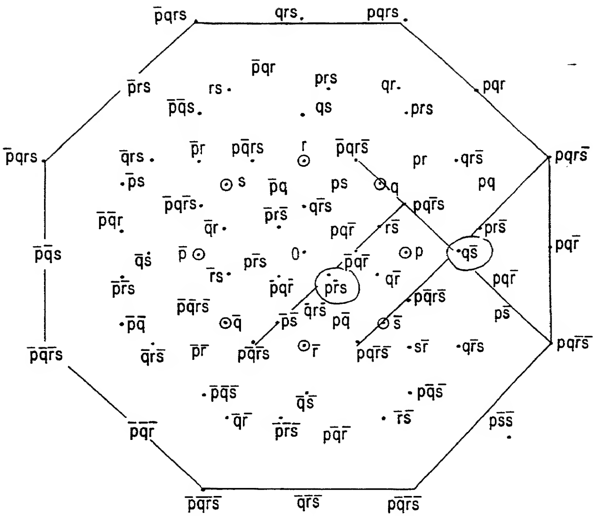

Figure 17

Four-Variable V-diagram with Simplification of
$pqr\bar{s}$ v $pq\bar{r}\bar{s}$ v $pq\bar{r}s$ v $\bar{p}qr\bar{s}$ v $p\bar{q}\bar{r}s$ v $\bar{p}q\bar{r}\bar{s}$.→. $p\bar{r}s$ v $q\bar{s}$

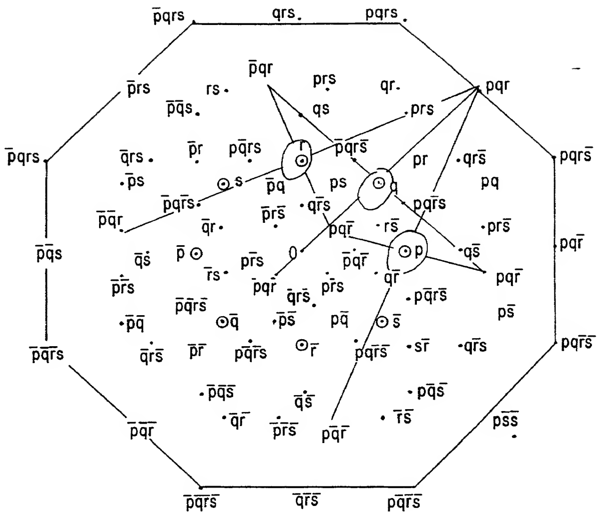

Figure 18A

Fix Rule d = 2

$pqr \vee pq\bar{r} \vee p\bar{q}r \vee p\bar{q}\bar{r} \vee \bar{p}qr \vee \bar{p}q\bar{r} \vee \bar{p}\bar{q}r . \rightarrow . \; p \vee q \vee r$

p	q	r	
1	1	-1	+
1	-1	-1	
1	0	0	

Figure 18B

p	q	r	
1	1	-1	+
1	-1	1	
1	0	0	

Figure 18C

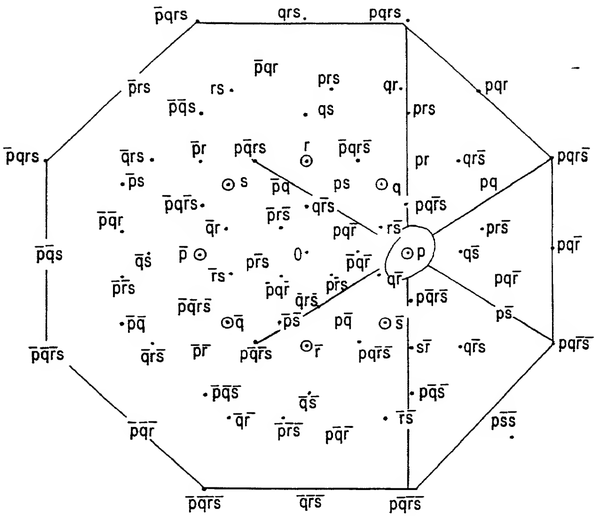

Figure 19

Fix Rule d = 3

$pqrs \vee pqr\bar{s} \vee pq\bar{r}s \vee pq\bar{r}\bar{s} \vee p\bar{q}rs \vee p\bar{q}r\bar{s} \vee p\bar{q}\bar{r}s \vee p\bar{q}\bar{r}\bar{s}.$ p

09/787290

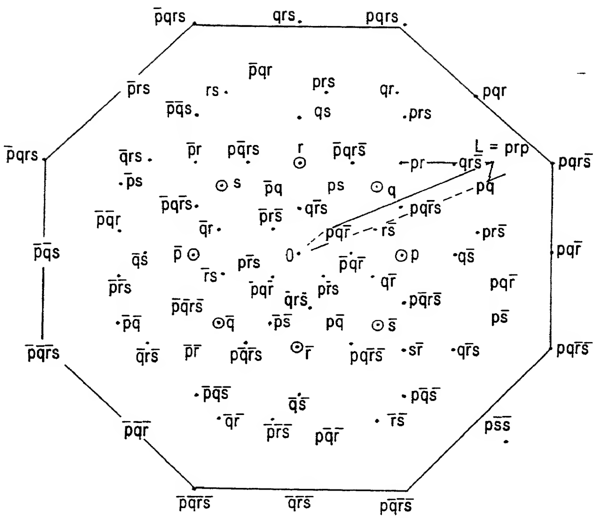

Figure 20

Undeveloped pq v p$\bar{q}$r v $\bar{p}\bar{q}\bar{r}$. . pq v pr v $\bar{p}\bar{q}\bar{r}$

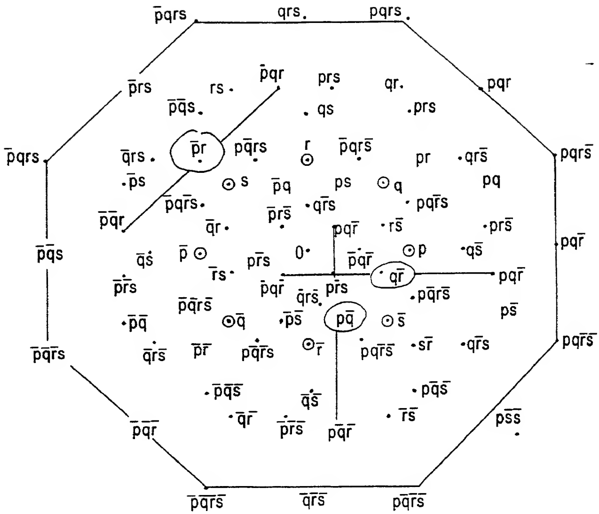

Figure 21

Simplification of Developed $\bar{p}q \vee p\bar{q} \vee \bar{q}r \vee q\bar{r}$:
$\bar{p}qr \vee \bar{p}\bar{q}r \vee \bar{p}q\bar{r} \vee pq\bar{r} \vee p\bar{q}\bar{r} \vee p\bar{q}r \therefore p\bar{q} \vee \bar{p}r \vee p\bar{r}$

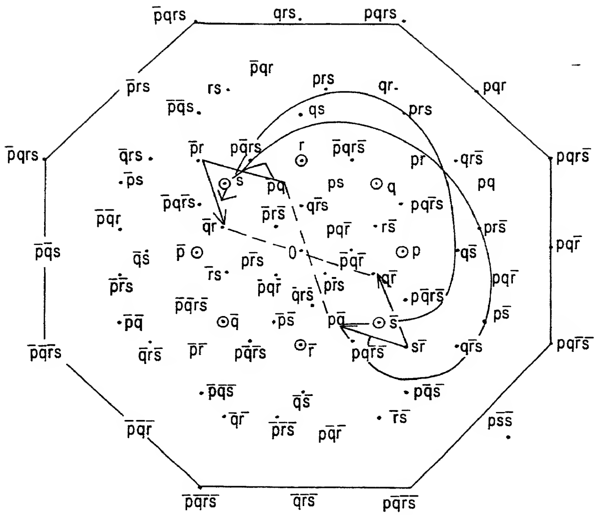

Figure 22

Simplification of Undeveloped $\bar{p}q$ v $\bar{q}r$ v $p\bar{q}$ v $q\bar{r}$

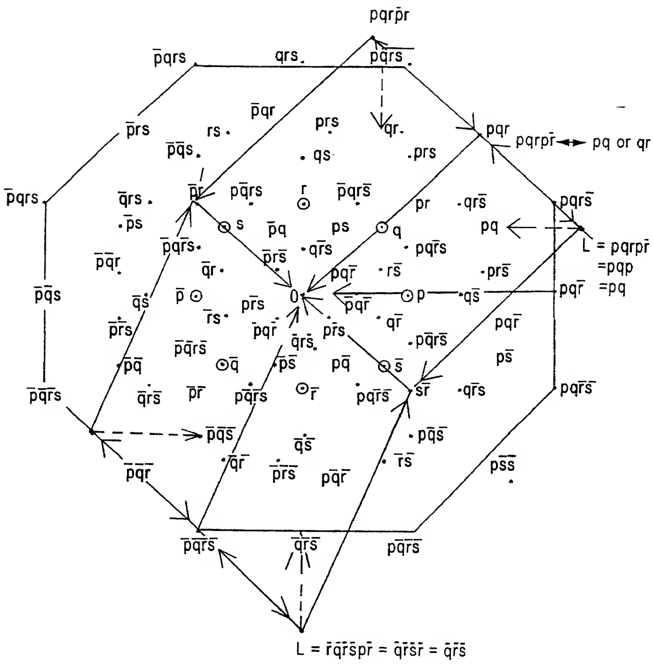

Figure 23

Simplification of $pqr \vee p\bar{r} \vee pq\bar{s} \vee \bar{p}r \vee \bar{p}\bar{q}\bar{r}\bar{s}$

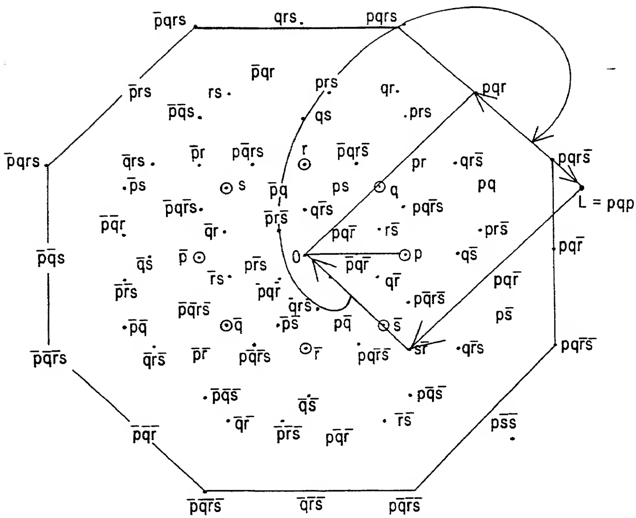

Figure 24

Equivalence of pqr and pq in pqr v p$\bar{r}$

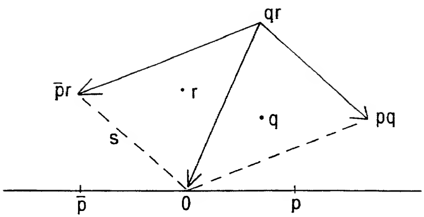

Figure 25
Consensus Theorem

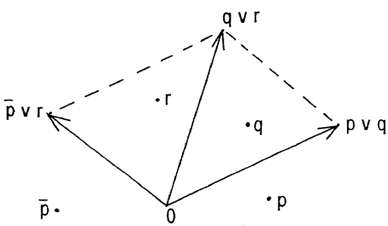

Figure 26
Consensus Theorem:
the Dual of qp v pr̄ v qr is (q v p)(p̄ v r)(q v r)

	ANS			CNS		
$p\ \bar{q}\ r$	qr v	pq v	$\bar{p}r$	$(q \vee p)$	$(\bar{p} \vee r)$	$(q \vee r)$
T T T	T	T				
T T F		T			F	
T F T						
T F F					F	F
F T T	T		T			
F T F						
F F T			T	F		
F F F				F		F

Figure 24A

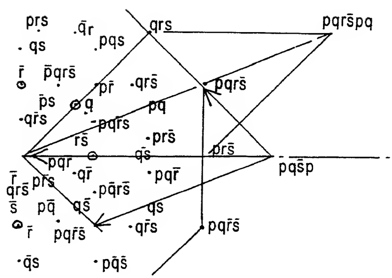

Figure 27

: Superfluity of $pq\bar{s}$ in $pqr \vee p\bar{r} \vee pq\bar{s}$

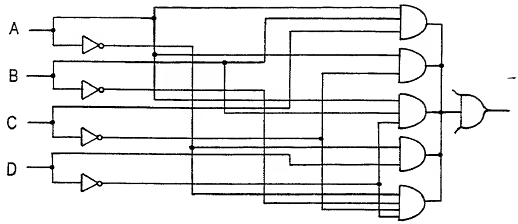

Figure 28

Target Circuit: $ABC + A\bar{C} + AB\bar{D} + \bar{A}C + \bar{A}\bar{B}\bar{C}\bar{D}$
G=5, I=12, R=7

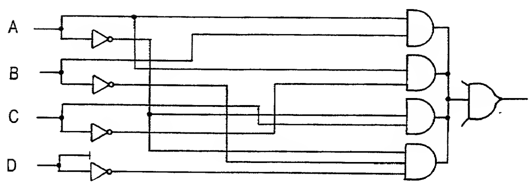

Figure 29

A Simplest Circuit Equivalent to the Target Circuit
$(ABC + A\bar{C} + ABD + \bar{A}C + \bar{A}\bar{B}\bar{C}\bar{D}) =$
$(AB + A\bar{C} + \bar{A}C + \bar{A}\bar{B}\bar{D})$
G=4, I=9, R=2

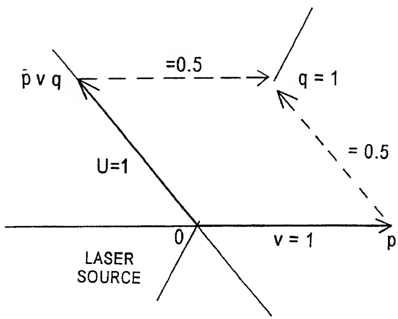

Figure 30

Optical Computation of modus ponens

⟶ Implication

- - -→ Half-implication

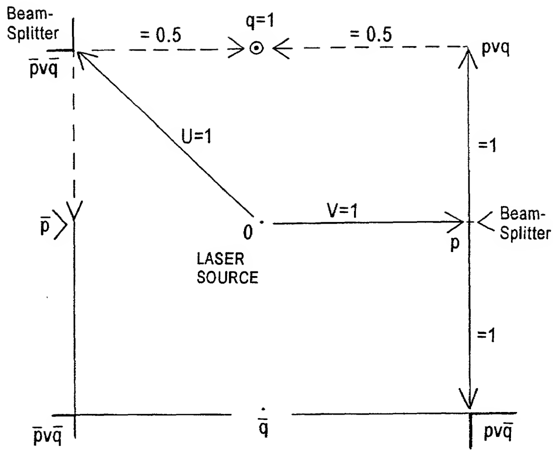

Figure 31
Interferometric Processing for modus ponens

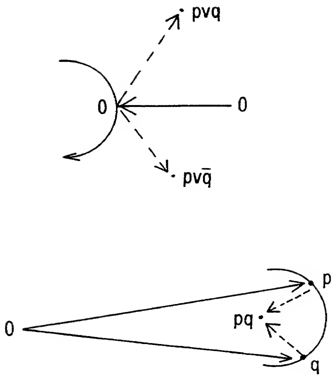

Figure 32

p and pq nodes in (p,q) space

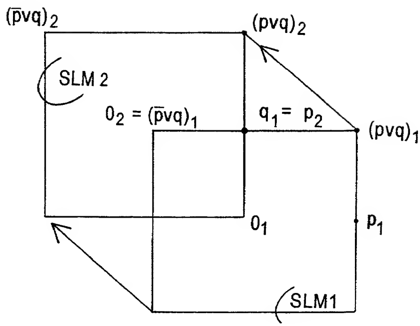

Figure 33

Displacement of O to $\bar{p}$ v q; p is q

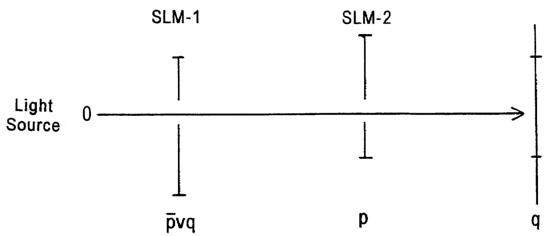

Figure 34
Vector Addition with Sequences of SLMs for modus ponens

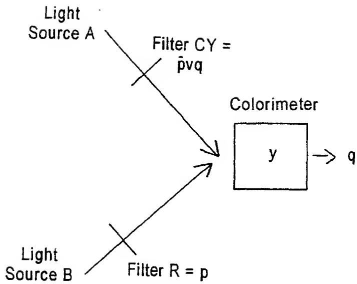

Figure 35

Colorimetric Computation of modus ponens

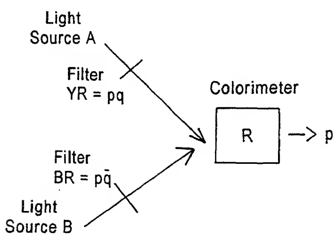

Figure 36

Colorimetric Simplification of pq v p$\bar{q}$

COMBINED DECLARATION FOR PATENT APPLICATION AND POWER OF ATTORNEY (Includes Reference to PCT International Application(s)	Attorney's Docket Number 52254-016

As below named inventor, I hereby declare that:

My residence, post office address and citizenship are as stated below next to my name,

I believe I am the original, first and sole inventor (if only one name is listed below) or an original, first and joint inventor (if plural names are listed below) of the subject matter which is claimed and for which a patent is sought on the invention entitled:

DEVICES AND TECHNIQUES FOR LOGICAL PROCESSING

the specification of which.

☐ is attached hereto.

☒ was filed as United States application Serial No. 09/787,290

on 15 March 2001

and was amended on ______ (if applicable).

☒ was filed as PCT international application Number PCT/US99/21955

on 22 September 1999

and was amended under PCT Article 19 on ______ (if applicable)

I hereby state that I have reviewed and understand the contents of the above-identified specification, including the claims, as amended by any amendment referred to above.

I acknowledge the duty to disclose information which is known to me to be material to patentability in accordance with Title 37, Code of Federal Regulations, §1.56.

I hereby claim foreign priority benefits under Title 35, United States Code, §119(a)-(d) or Section 365(b) of any foreign and/or international application(s) for patent or inventor's certificate or Section 365(a) of any PCT international application(s) designating at least one country other than the United States of America listed below and have also identified below any foreign application(s) for patent or inventor's certificate or any PCT international application(s) designating at least one country other than the United States of America filed by me on the same subject matter having a filing date before that of the application(s) of which priority is claimed:

PRIOR FOREIGN/PCT APPLICATION(S) AND ANY PRIORITY CLAIMS UNDER 35 U.S.C. 119:

COUNTRY (If PCT, indicate "PCT")	APPLICATION NUMBER	DATE OF FILING (day, month, year)	PRIORITY CLAIMED UNDER 35 USC 119
PCT	PCT/US99/21955	22 September 1999	☒Yes ☐ No

I hereby claim the benefit under 35 USC §119(e) of any United States provisional application(s) listed below.

PRIOR PROVISIONAL APPLICATION(S):

Application Number	Filing Date
60/101,371	22 September 1998

I hereby claim the benefit under Title 35, United States Code, §120 of any United States application(s), or §365(c) of any PCT international application(s) designating the United States of America that is/are listed below and, insofar as the subject matter of each of the claims of this application is not disclosed in that/those prior application(s) in the manner provided by the first paragraph of Title 35, United States Code, §112, I acknowledge the duty to disclose information which is material to patentability as defined in Title 37, Code of Federal Regulations, §1.56 which occurred between the filing date of the prior application(s) and the national or PCT international filing date of this application.

PRIOR U.S. APPLICATIONS OR PCT INTERNATIONAL APPLICATIONS DESIGNATING THE U.S. FOR BENEFIT UNDER 35 U.S.C. 120:

U.S. APPLICATIONS			STATUS (Check One)		
U.S. Application Number	U.S Filing Date		Patented	Pending	Abandoned
PCT APPLICATIONS DESIGNATING THE U.S.					
PCT Application No.	PCT Filing Date	U.S. Serial Numbers Assigned *(if any)*			
PCT/US99/21955	22 September 1999	09/787,290			

POWER OF ATTORNEY: As named inventor, I hereby appoint the following attorney(s) and/or agent(s) to prosecute this application and transact all business in the Patent and Trademark Office connected therewith: Stephen A Becker, Reg. No. 26,527; John G. Bisbikis, Reg. No. 37,095; Christopher D. Bright, Reg. No. 46,578; Daniel Bucca, Reg. No. 42,368; Kenneth L. Cage, Reg. No. 26,151; Jennifer Chen, Reg. No. 42,404; Bernard P. Codd, Reg. No 46,429; Lawrence T. Cullen, Reg. No. 44,489; Paul Devinsky, Reg. No. 28,553; Margaret M. Duncan, Reg. No. 30,879; Shamita De. Etienne-Cummings, Reg. No. 46,072; Ramyar M. Farid, Reg. No 46,692; Brian E. Ferguson, Reg. No. 36,801; Michael E. Fogarty, Reg. No. 36,139; John R. Fuisz, Reg. No. 37,327; Willem F. Gadiano, Reg. No. 37,136; Keith E. George, Reg. No. 34,111; Matthew V. Grumbling, Reg. No. 44,427; John A. Hankins, Reg. No. 32,029; Eric J. Kraus, Reg. No. 36,190; Catherine Krupka, Reg. No. 46,227; Jack Q. Lever, Reg. No. 28,149; Raphael V. Lupo, Reg. No. 28,363; Burman Y. Mathis III, Reg. No. 44,907; Michael A. Messina, Reg. No. 33,424; Dawn L. Palmer, Reg. No. 41,238; Joseph H. Paquin, Jr., Reg. No. 31,647; Scott D. Paul, Reg. No. 42,984; William D. Pegg, Reg. No. 42,988; Robert L. Price, Reg. No. 22,685; Gene Z. Rubinson, Reg. No. 33,351; Joy Ann G. Serauskas, Reg. No. 27,952; Daniel H. Sherr, Reg No. 46,425; David A. Spenard, Reg. No. 37,449; Arthur J. Steiner, Reg. No. 26,106; David L. Stewart, Reg. No. 37,578; Wesley Strickland, Reg. No. 44,363; Michael D. Switzer, Reg. No. 39,552; Daniel S. Trainor, Reg. No. 43,959; Cameron K. Weiffenbach, Reg. No. 44,488; Aaron Weisstuch, Reg. No. 41,557; Edward J. Wise, Reg. No. 34,523; Jeffrey A. Woller, Reg. No. 48,041; Alexander V. Yampolsky, Reg. No. 36,324; and Robert W. Zelnick, Reg No. 36,976 , all of McDermott, Will & Emery.

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I hereby declare that all statement made herein of my own knowledge are true and that all statements made on information and belief are believed to be true; and further that these statements were made with the knowledge that willful false statements and the like so made are punishable by fine or imprisonment, or both, under section 1001 of Title 18 of the United States Code, and that such willful false statements may jeopardize the validity of the application or any patent issuing thereon.

Signature of Inventor 201:	**Signature of Inventor 202:**	**Signature of Inventor 203:**
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Date June 9 2001	**Date**	**Date**